

This notebook demonstrates some of the behavior of the ODEs you'll be studying during Project 5, on trajectories of rockets in a changing gravitational field.

In recitation, we'll find that the differential equation of motion for a rocket starting at the surface of a planet with radius R and mass M with some velocity v_0 is given by

$$\dot{y}[t]^2 = -\frac{2GM}{R} + v_0^2 + \frac{2GM}{R+y[t]}$$

The first term, $2GM/R$, we also showed was the square of the escape velocity. The behavior of a solution is going to depend on how the initial velocity is related to the escape velocity.

In general, we'll have specific values for the constants involved. For example, with the Earth, R is roughly 6400 km and M is about 5.8×10^{24} kg; G is the universal gravitational constant, which is about 6.67×10^{-11} N m² / kg². This leads to an escape velocity on the order of 11.2 km/s, or 25,000 miles per hour.

For the examples in this notebook, I'll simplify things a bit by choosing scales where the numbers are significantly nicer. In fact, I'll take $G = 2$, $M = 1$ and $R = 1$ for demonstration; in this case, the escape velocity is 2.

We'll come up with the plots of three behaviors: At, above, and below escape velocity.

(* At escape velocity, where y reduces to a separable equation! *)

$$(* y'[t]^2 = 2GM / (R + y[t]) = 4 / (1 + y[t]) *)$$

```
In[20]:= escape = DSolve[{y'[t] == Sqrt[4 / (1 + y[t])], y[0] == 0}, y[t], t]
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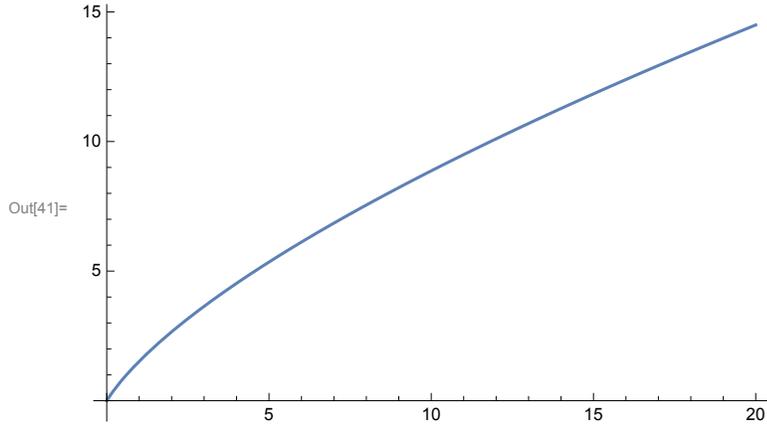
DSolve::bvnul : For some branches of the general solution, the given boundary conditions lead to an empty solution. >>

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```
Out[20]= {{y[t] -> -1 + ((-1 + 3 t)^2)^(1/3)}, {y[t] -> -1 + ((1 + 3 t)^2)^(1/3)}}
```

(* Notice that we have two branches, corresponding to the two sign choices in the square root. The second one is the correct one. *)

```
In[41]:= plotEscape = Plot[-1 + ((1 + 3 t)^2)^{1/3}, {t, 0, 20}]
```

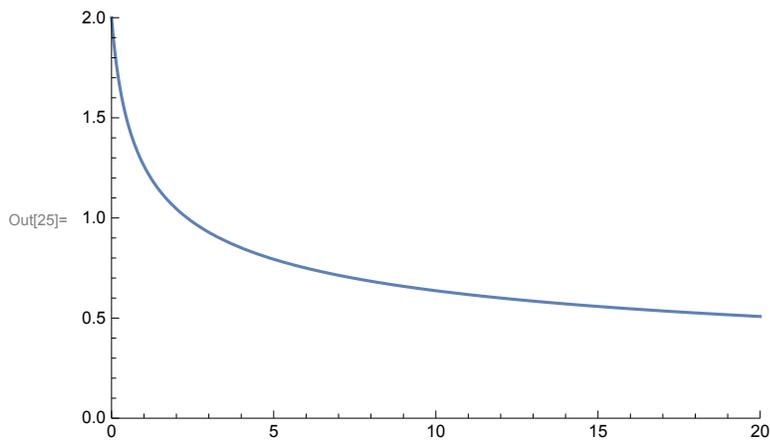


```
In[23]:= velocityEscape = D[-1 + ((1 + 3 t)^2)^{1/3}, t]
```

Out[23]=

$$\frac{2 (1 + 3 t)}{(1 + 3 t)^{2/3}}$$

```
In[25]:= plotVelocityEscape = Plot[velocityEscape, {t, 0, 20}, PlotRange -> {{0, 20}, {0, 2}}]
```



So we see what we'd expect: Velocity slowly tending towards zero as height tends to infinity.

In[36]=

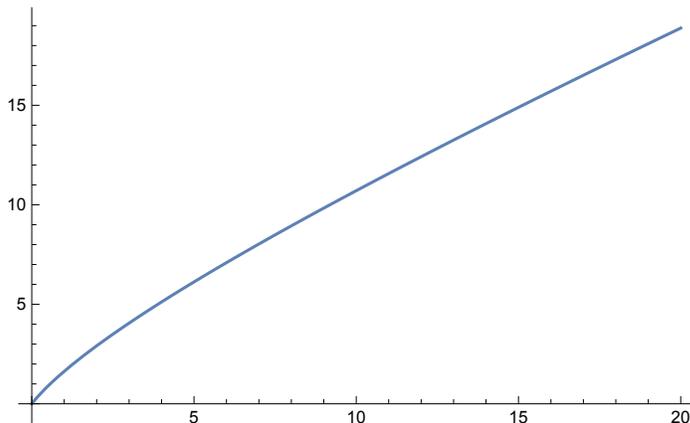
```
(* Now let's look at the behavior when
the initial velocity is a little above the escape,
say 2.1 = 21/10. I'll use the fraction because Mathematica
does better with exact numbers here. *)
aboveEscape = DSolve[{y'[t] == Sqrt[(21/10)^2 - 2^2 + 4 / (1 + y[t])], y[0] == 0},
y, t]
```

```
Out[36]= {{y -> Function[{t}, InverseFunction[
(-400 Sqrt[41] ArcSinh[1/20 Sqrt[41] Sqrt[1 + #1]] Sqrt[441 + 41 #1] + 41 Sqrt[1 + #1] (441 + 41 #1)) /
(1681 Sqrt[1 + #1] Sqrt[441 + 41 #1] / (1 + #1)) &][t/10 + (861 - 400 Sqrt[41] ArcSinh[1/20 Sqrt[41]]) / 1681]]}}
```

In[42]= plotAbove = Plot[InverseFunction[

```
(-400 Sqrt[41] ArcSinh[1/20 Sqrt[41] Sqrt[1 + #1]] Sqrt[441 + 41 #1] + 41 Sqrt[1 + #1] (441 + 41 #1)) /
(1681 Sqrt[1 + #1] Sqrt[441 + 41 #1] / (1 + #1)) &][t/10 + (861 - 400 Sqrt[41] ArcSinh[1/20 Sqrt[41]]) / 1681], {t, 0, 20}]
```

Out[42]=

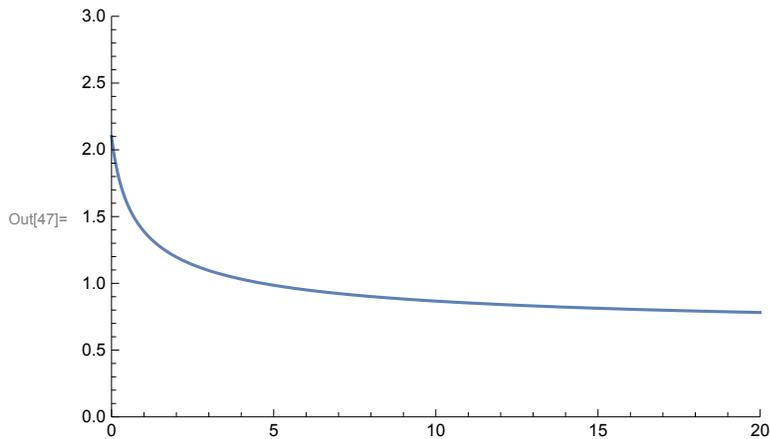


velocityAbove = D[InverseFunction[

```
(-400 Sqrt[41] ArcSinh[1/20 Sqrt[41] Sqrt[1 + #1]] Sqrt[441 + 41 #1] + 41 Sqrt[1 + #1] (441 + 41 #1)) /
(1681 Sqrt[1 + #1] Sqrt[441 + 41 #1] / (1 + #1)) &][t/10 + (861 - 400 Sqrt[41] ArcSinh[1/20 Sqrt[41]]) / 1681], t];
```

```
(* I'm suppressing the output because it's pretty bad. *)
```

```
In[47]:= plotVelocityAbove = Plot[velocityAbove, {t, 0, 20}, PlotRange -> {{0, 20}, {0, 3}}]
```



What we see now also makes sense. Our height is tending off to infinity and our velocity seems to be leveling out at a constant, and non-zero value. Our height is also appreciably higher than the escape velocity case, since we're moving a little faster at the outset. The difference gets really apparent if we run it for, say, 50 times units instead of 20. Just be warned that the plotting takes a pretty long time now.

```
In[48]:= (* Finally, let's take an initial velocity
a little below the escape velocity, say 1. *)
belowEscape = DSolve[{y'[t] == Sqrt[-4 + 1^2 + 4 / (1 + y[t])]}, y[0] == 0], y, t]
```

Out[48]=

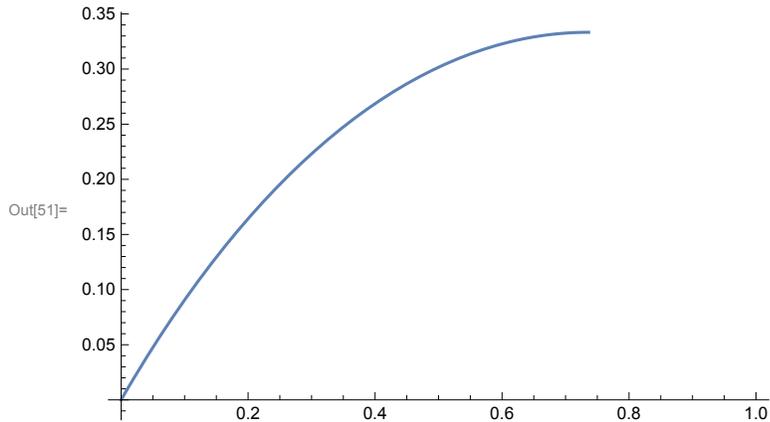
$$\left\{ \left\{ y \rightarrow \text{Function} \left[\{t\}, \right. \right. \right.$$

$$\left. \left. \text{InverseFunction} \left[\left(4 \text{ArcSin} \left[\frac{1}{2} \sqrt{3} \sqrt{1+\#1} \right] \sqrt{3-9\#1} + 3 \sqrt{1+\#1} (-1+3\#1) \right) / \right. \right. \right.$$

$$\left. \left. \left(9 \sqrt{\frac{1-3\#1}{1+\#1}} \sqrt{1+\#1} \right) \& \right] \left[\frac{1}{27} (-9+4\sqrt{3}\pi) + t \right] \right\} \right\}$$

In[51]= plotBelow =

$$\text{Plot}\left[\text{InverseFunction}\left[\left(4 \text{ArcSin}\left[\frac{1}{2} \sqrt{3} \sqrt{1+\#1}\right] \sqrt{3-9\#1} + 3 \sqrt{1+\#1} (-1+3\#1)\right)\right] / \left(9 \sqrt{\frac{1-3\#1}{1+\#1}} \sqrt{1+\#1}\right) \&]\left[\frac{1}{27} (-9+4\sqrt{3}\pi) + t\right], \{t, 0, 1\}]$$



In[52]= (* Notice how the plot just stops at around t =

0.7 and height 0.33. This is when our object reverses and starts falling back to the planet. Since we took the positive branch of the square root, this is expected behavior: Our solution's domain stops at this time. *)

(* Let's look at a case where we were a little closer to escape velocity, say $y[0] = 1.9 = 19/10$. *)

belowEscapeFaster =

$$\text{DSolve}\left[\{y'[t] == \text{Sqrt}[-4 + (19/10)^2 + 4 / (1 + y[t])], y[0] == 0\}, y, t\right]$$

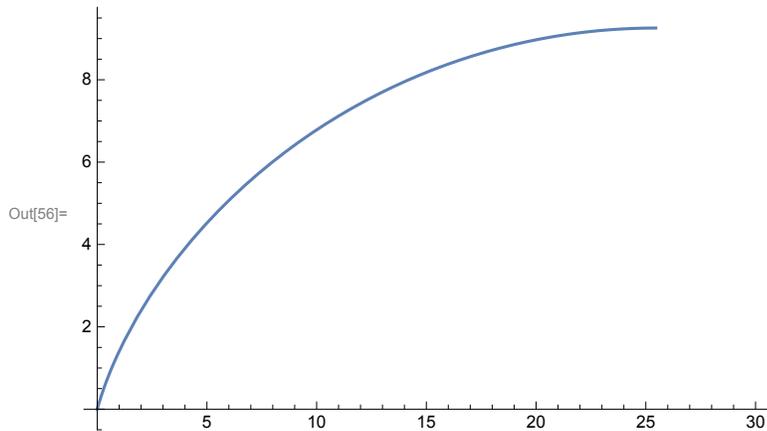
Out[52]=

$$\left\{\left\{y \rightarrow \text{Function}\left[\{t\}, \text{InverseFunction}\left[\frac{400 \sqrt{39} \text{ArcSin}\left[\frac{1}{20} \sqrt{39} \sqrt{1+\#1}\right] \sqrt{361-39\#1} + 39 \sqrt{1+\#1} (-361+39\#1)}{1521 \sqrt{\frac{361-39\#1}{1+\#1}} \sqrt{1+\#1}}\right] \&]\left[\frac{t}{10} + \frac{-741 + 400 \sqrt{39} \text{ArcSin}\left[\frac{\sqrt{39}}{20}\right]}{1521}\right]\right]\right\}\right\}$$

```
In[56]:= plotBelowFaster = Plot[InverseFunction[
  
$$\frac{400 \sqrt{39} \operatorname{ArcSin}\left[\frac{1}{20} \sqrt{39} \sqrt{1+\#1}\right] \sqrt{361-39\#1} + 39 \sqrt{1+\#1} (-361+39\#1)}{1521 \sqrt{\frac{361-39\#1}{1+\#1}} \sqrt{1+\#1}}$$

  &][
  
$$\frac{t}{10} + \frac{-741 + 400 \sqrt{39} \operatorname{ArcSin}\left[\frac{\sqrt{39}}{20}\right]}{1521}$$

], {t, 0, 30}]
```



(* Now it takes far longer, about 9 time units,
for the rocket to level out. It also gets to a much greater maximum height. *)